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Pfaff systems with prescribed stability group ★

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Abstract

A method to construct Pfaff Systems whose stability group contains a prescribed Lie group acting transitively is given. All of such Pfaff systems can be obtained by this method. © 1999 Elsevier Science B.V. All right reserved

1. Introduction

Let M be a differentiable manifold and $\omega = \{\omega^1, \dots, \omega^r\}$ a set composed by r differential 1-forms on M. We assume that the dimension of the vector space generated by the values of the elements of ω at any point is independent of that point.

A diffeomorphism, φ, defined in an open subset, U, of M is said to stabilize ω, if there exist functions $\varphi^i_j \in C^\infty(U)$ such that

$$\varphi^* \omega^i = \sum_{j=1}^{r} \varphi^i_j \omega^j$$

on U, for all $i = 1, \dots, r$.

We denote by $\mathrm{Est}(\omega)$ the set composed by the global diffeomorphisms of M that stabilizes ω. Actually $\mathrm{Est}(\omega)$ is a group under composition of diffeomorphisms, which is called the *stability group* of ω.

The stability group is not, in general, realizable as a (finite dimensional) Lie group acting on M. In some cases it reduces to the identity element, but in other cases it coincides with the whole diffeomorphism group.

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Nevertheless, one can consider heuristically as its Lie algebra the set, est(ω), composed by the vector fields whose flow is composed by diffeomorphisms that stabilizes ω. Thus est(ω) is composed by the vector fields, X, on M such that

$$L_X\omega^i = \sum_{j=1}^{r} \psi^i_j \omega^j$$

for all $i = 1, \dots, r$, where $\psi^i_j \in C^\infty(M)$. The set est(ω) is easily seen to be a Lie algebra under ordinary Lie bracket, and is called the *infinitesimal stability algebra* of ω.

There are two problems related to stability groups which are of some interest:

(a) Given a Pfaff system to determine its stability group, its infinitesimal stability algebra and some of its properties. One of the properties to be studied is the existence of subgroups of the stability group, which are realizable as Lie groups acting on M. This is a problem directly related to the search for finite dimensional Lie subalgebras of the infinitesimal stability algebra. For example, some properties of the infinitesimal stability algebra in the case of a contact form have been studied in [12].
(b) Given a Lie group, G, to find a differentiable manifold, M, a Pfaff system on M, ω, and an action of G on M by elements of the stability group of ω. This is a classical way to realize a Lie group. For example, Cartan and Engel have represented the exceptional Lie groups in this way (cf. [2] (reprinted in [4, pp. 107–132]), [3] (reprinted in [4, pp. 137–287]) and [9]).

The present paper is related to problem (b). In fact we give all solutions corresponding to the case where the action is transitive, although this is not the case of the cited examples of Cartan and Engel.

Let us denote by S_p the vector subspace of T^*_pM generated by the values at $p \in M$ of $\omega^1, \dots, \omega^r$. These subspaces compose a vector subbundle of T^*M, S. The fact that a Lie group, G, acts on M by elements of Est(ω) is equivalent to the fact that the canonical lift of this action to T^*M preserves S (under these circumstances we say that the action of G stabilizes S).

Conversely, let M be a homogeneous space of G and S a vector subbundle T^*M such that the action of G stabilizes S. Let φ be one of the diffeomorphisms associated to elements of G by the action and $\{\omega^1_a, \dots, \omega^r_a\}$ be a local section of the principal fibre bundle of the basis of S defined on an open subset of M, U_a, $a = 1, 2$, such that $\varphi(U_1) = U_2$, then there exist functions $\varphi^i_j \in C^\infty(U_1)$ such that

$$\varphi^*\omega^i_2 = \sum_{j=1}^{r} \varphi^i_j \omega^j_1$$

for all $i = 1, \dots, r$. In this sense, the pair (M, S) is a "local" solution to our problem. The solutions to our problem corresponds in the strict sense to the case where this principal fibre bundle has a global cross-section, i.e. it is trivial.

In Section 4, a method is given to construct local solutions in the aforementioned sense.

In Section 5, sufficient conditions are given for the existence of global sections of the principal fibre bundle of basis of the constructed vector bundle.

The results obtained in Section 3, most of them valid in the non-transitive case, gives us motivation for the other sections and proves that the construction we made at Section 4 gives all possible local solutions.

Finally, let us say some words about the precedents of this paper.

The subset of Est(ω) composed by the elements, φ, such that $\varphi^*\omega^i = \omega^i$ for all $i = 1, \ldots, r$, compose a subgroup which we shall denote by Inv(ω).

The case where Inv(ω) contains a transitive Lie subgroup has been studied in [7], where the following problem has been solved: if G is a Lie group, to find all homogeneous spaces of G, M, and Pfaff systems on M, ω, with vanishing characteristic system, such that the diffeomorphisms corresponding to the action are in Inv(ω). These Pfaff systems give rise to principal fibre bundles with connection, whose structural group is abelian. Particular cases are the homogeneous contact manifolds, which have been studied in many papers. The most celebrated is perhaps that of Boothby and Wang [1] in which the authors prove that a simply connected compact homogeneous contact manifold is the total space of a principal circle bundle over a simply connected Hodge manifold, where the contact form defines a connection. They also prove that when a simply connected Hodge manifold is given, there exists a principal circle bundle on it whose total space is a regular contact compact manifold. These results were slightly improved in [8].

Generalization of these results to non-compact homogeneous contact manifolds and ways to explicitly construct the homogeneous contact manifolds and the corresponding principal bundles can be found in [5–7,14]. These fibre bundles with connection given by a contact form are the starting point of geometric quantization (Kirillov–Kostant–Souriau theory [11,13]).

2. Notation

General notations concerning differentiable manifolds, tensor fields and fibre bundles are as in [10], unless otherwise stated.

If G is a Lie group, we denote by $\underline{G}$ the Lie algebra of left invariant vector fields on G and by $\underline{G}^*$ the vector space of left invariant 1-forms on G. The identity element of G is denoted by e.

Given $X \in \underline{G}$, we denote by X_d the right invariant vector field whose value at e coincides with that of X.

Let us assume that G acts on the left (resp. right) on a differentiable manifold, M. If $g \in G$, we denote by g_M the corresponding diffeomorphism of M. For all $X \in \underline{G}$ we denote by X_M the vector field on M whose flow is given by $\{\mathrm{Exp}(-tX)_M\colon t \in \mathbb{R}\}$ (resp. $\{\mathrm{Exp}(tX)_M\colon t \in \mathbb{R}\}$). With this notation the map defined by sending each $X \in \underline{G}$ to X_M is a Lie algebra homomorphism.

If $\gamma(t)$ is a curve in a manifold, we denote by $\overline{\dot{\gamma}(t)}$ its tangent vector at $t = 0$, unless otherwise stated.

3. Homogeneous Pfaff systems whose stability group contains a Lie subgroup

In this paper we call Pfaff system of rank r each pair (ρ, λ), where $\rho : S \mapsto M$ is a vector bundle of rank r and $\lambda : S \mapsto T^*M$ is an injective homomorphism of vector bundles over the identity of M.

We identify S with $\lambda(S)$ by means of λ. Thus a local cross-section of ρ can be considered as a 1-form on an open subset of M. The map ρ becomes the restriction of τ^* to S, where τ^* is the canonical map from T^*M onto M, and λ becomes the canonical injection of S into T^*M.

Let φ be a diffeomorphism of M. As usual, we denote by $(\varphi^{-1})^*$ the diffeomorphism of T^*M given by

$$((\varphi^{-1})^*\alpha)(v) = \alpha(T_{\varphi(x)}\varphi^{-1}(v))$$

for all $x \in M$, $\alpha \in T_x^*M$ and $v \in T_{\varphi(x)}M$.

We say that φ stabilizes S if $(\varphi^{-1})^*(S) = S$.

The set composed by the diffeomorphisms of M that stabilizes S is a group under the usual composition of maps that is called the stability group of S.

We denote by $BS(M, GL(r, \mathbb{R}))$ the principal fibre bundle of the basis of S.

BS is composed by $(\alpha^1, \dots, \alpha^r)$ such that the α^i are linearly independent elements of some fibre of ρ. If U is an open set of M and $\omega^1, \dots, \omega^r$ are differentiable sections of ρ defined in U whose values at each point are linearly independent, then

$$(\omega^1, \dots, \omega^r) : x \in U \mapsto (\omega^1(x), \dots, \omega^r(x))$$

is a differentiable section of this principal fibre bundle, which is denoted by $(U, (\omega^1, \dots, \omega^r))$.

The bundle action of $GL(r, \mathbb{R})$ on the right on BS is given by

$$(\alpha^1, \dots, \alpha^r) \star L = (\beta^1, \dots, \beta^r),$$

where

$$\beta^i = \sum_{j=1}^{r} L_i^j \alpha^j,$$

$i = 1, \dots, r$, L_i^j being the element of L in row j, column i $((j, i)$ entry).

The bundle projection is denoted by $B\rho : BS \to M$. We have $B\rho(\alpha^1, \dots, \alpha^r) = x$ if $\rho(\alpha^i) = x$, $i = 1, \dots, r$.

Let φ be a diffeomorphism of M that stabilizes S. We define

$$\varphi_{BS} : (\alpha^1, \dots, \alpha^r) \in BS \to ((\varphi^{-1})^*\alpha^1, \dots, (\varphi^{-1})^*\alpha^r) \in BS.$$

φ_{BS} is an automorphism of the principal fibre bundle $BS(M; GL(r, \mathbb{R}))$, i.e. it is a diffeomorphism of BS that commutes with the bundle action.

The following lemma is more or less obvious and is equivalent to a result that has been used in Section 1.

Lemma 3.1. *Let φ be a diffeomorphism of M. The following conditions are equivalent:*
(i) φ stabilizes S.
(ii) For all sections $(U_a, (\omega_a^1, \dots, \omega_a^r))$, $a = 1, 2$, such that $\varphi(U_1) = U_2$, there exist functions $F_j^i \in C^\infty(U_2)$ such that

$$(\varphi^{-1})^* \omega_1^i = \sum_{j=1}^{r} F_j^i \omega_2^j$$

for all $i = 1, \dots, r$.

Let G be a Lie group acting on the left on M by diffeomorphisms that stabilizes S. These diffeomorphisms compose a subgroup of the stability group that admits a Lie group structure: it is isomorphic to the quotient of G by the closed normal subgroup of G composed by the elements whose corresponding diffeomorphisms are the identity. Under these circumstances, we say that G is a *stability subgroup* of S. If the action is transitive we say that G is a *transitive stability subgroup* of S.

The map defined by sending each $g \in G$ to the diffeomorphism $(g_M^{-1})^*$ of T^*M is an action on the left and so is the map that associates to each $g \in G$ the restriction of $(g_M^{-1})^*$ to S (differentiability of the action follows from the fact that λ is an embedding). In a similar way, the map defined by sending $g \in G$ to $(g_M)_{BS}$ is an action on the left on BS. In what follows, we denote $(g_M)_{BS}$ simply by g_{BS} and the restriction of $(g_M^{-1})^*$ to S by g_S.

In BS the *canonical 1-form with values in* $\mathbb{R}^r$, Ω, is defined by means of

$$\Omega_{(\alpha^1,\dots,\alpha^r)} \cdot v = (\alpha^1(T_{(\alpha^1,\dots,\alpha^r)} B\rho \cdot v), \dots, \alpha^r(T_{(\alpha^1,\dots,\alpha^r)} B\rho \cdot v))$$

for all $(\alpha^1, \dots, \alpha^r) \in BS$, $v \in T_{(\alpha^1,\dots,\alpha^r)} BS$.

If $(\omega^1, \dots, \omega^r)$ is a differentiable cross-section of $B\rho$ defined in an open subset, U, of M, we define differentiable functions in $B\rho^{-1}(U)$, ω_j^i, by means of

$$\sum_{j=1}^{r} \omega_j^i(\alpha^1, \dots, \alpha^r) \omega^j_{B\rho(\alpha^1,\dots,\alpha^r)} = \alpha^i$$

for all $(\alpha^1, \dots, \alpha^r) \in B\rho^{-1}(U)$.

Since the $\omega_j^i(\alpha^1, \dots, \alpha^r)$ are the components of the elements of a basis in another basis, the matrix whose entries are $\omega_j^i(\alpha^1, \dots, \alpha^r)$ is non-singular. In the next pages we will refer to this fact by saying that (ω_j^i) takes its values in $GL(r, \mathbb{R})$.

Lemma 3.2. *In $B\rho^{-1}(U)$ we have*

$$\Omega = \left(\sum_{j=1}^{r} \omega_j^1 (B\rho)^* \omega^j, \dots, \sum_{j=1}^{r} \omega_j^r \, (B\rho)^* \omega^j \right).$$

Proof. Let $(\alpha^1, \ldots, \alpha^r) \in B\rho^{-1}(U)$ and $v \in T_{(\alpha^1,\ldots,\alpha^r)}BS$. Then, for all $i = 1, \ldots, r$, we have

$$\left(\sum_{j=1}^{r} \omega_j^i (B\rho)^* \omega^j\right)_{(\alpha^1,\ldots,\alpha^r)} \cdot v = \sum_{j=1}^{r} \omega_j^i(\alpha^1, \ldots, \alpha^r)\omega^j_{B\rho(\alpha^1,\ldots,\alpha^r)}(T_{(\alpha^1,\ldots,\alpha^r)}B\rho \cdot v)$$

$$= \alpha^i (T_{(\alpha^1,\ldots,\alpha^r)}B\rho \cdot v)$$

and the result follows. □

In the following we denote by $\Omega^1, \ldots, \Omega^r$ the components of Ω.

Proposition 3.3. *Let Λ be a diffeomorphism of M that stabilizes S. Then $\Lambda^*_{BS}\Omega = \Omega$.*

Proof. If $(\alpha^1, \ldots, \alpha^r) \in BS$ and $v \in T_{(\alpha^1,\ldots,\alpha^r)}BS$ we have

$$\begin{aligned}
(\Lambda^*_{BS}\Omega^i)_{(\alpha^1,\ldots,\alpha^r)} \cdot v &= \Omega^i_{\Lambda_{BS}(\alpha^1,\ldots,\alpha^r)}(T_{(\alpha^1,\ldots,\alpha^r)}\Lambda_{BS} \cdot v) \\
&= \Omega^i_{((\Lambda^{-1})^*\alpha^1,\ldots,(\Lambda^{-1})^*\alpha^r)}(T_{(\alpha^1,\ldots,\alpha^r)}\Lambda_{BS} \cdot v) \\
&= (\Lambda^{-1})^*\alpha^i (T_{((\Lambda^{-1})^*\alpha^1,\ldots,(\Lambda^{-1})^*\alpha^r)}B\rho \circ T_{(\alpha^1,\ldots,\alpha^r)}\Lambda_{BS} \cdot v) \\
&= ((\Lambda^{-1})^*\alpha^i (T_{(\alpha^1,\ldots,\alpha^r)}(\Lambda \circ B\rho) \cdot v) = \alpha^i (T_{(\alpha^1,\ldots,\alpha^r)}B\rho \cdot v) \\
&= \Omega^i_{(\alpha^1,\ldots,\alpha^r)} \cdot v. \qquad \square
\end{aligned}$$

Let $A = (\alpha^1, \ldots, \alpha^r) \in BS$ and $a = B\rho(A)$. We denote by G_A (resp. G_a) the isotropy subgroup at A (resp. a) of the action of G on BS (resp. M).

If $g \in G_a$, $g_{BS}(A)$ is also a basis of the fibre of S at a, so that there exists an unique $s(g) \in GL(r; \mathbb{R})$ such that $g_{BS}(A) = A \star s(g)$. As a consequence of the fact that the action of G commutes with the bundle action, one sees that the map s defined by sending $g \in G_a$ to $s(g)$ is a representation of G_a. G_A is the kernel of this representation so that it is an invariant subgroup of G_a, and G_a/G_A is isomorphic to $s(G_a)$.

Let $\underline{a}$ (resp. $\underline{A}$) be the map from G onto M (resp. BS) defined by sending g to $g_M(a)$ (resp. $g_{BS}(A)$).

As a consequence of Preposition 3.3 the 1-forms Ω^i are invariant under the action of G so that the forms $\underline{A}^*\Omega^i$ are left invariant 1-forms on G. The subset of $\underline{G}^*$ composed by these 1-forms is denoted by Q. Now we introduce some notation concerning subsets of $\underline{G}^*$.

Let P be any subset of $\underline{G}^*$ and $\langle P \rangle$ the vector subspace it generates. We define:

$$\begin{aligned}
G_P &= \{g \in G\colon \mathrm{Ad}^*_g\alpha = \alpha,\ \forall\alpha \in P\} \\
F_P &= \{g \in G\colon \mathrm{Ad}^*_g\alpha \in \langle P \rangle,\ \forall\alpha \in P\} \\
\underline{G}_P &= \{X \in \underline{G}\colon i_X\,\mathrm{d}\alpha = 0,\ \forall\alpha \in P\} \\
\underline{F}_P &= \{X \in \underline{G}\colon i_X\,\mathrm{d}\alpha \in \langle P \rangle,\ \forall\alpha \in P\}
\end{aligned}$$

$$\underline{N}_P = \{X \in \underline{G}_P \colon i_X \alpha = 0,\ \forall \alpha \in P\}$$

$$\underline{H}_P = \{X \in \underline{F}_P \colon i_X \alpha = 0,\ \forall \alpha \in P\}$$

We obviously have $\underline{N}_P \subset \underline{G}_P \subset \underline{F}_P$ and $\underline{N}_P \subset \underline{H}_P \subset \underline{F}_P$.

Proposition 3.4. *The sets $\underline{N}_P, \underline{H}_P$, $\underline{G}_P$ and $\underline{F}_P$ are Lie subalgebras of $\underline{G}$, moreover, $\underline{N}_P, \underline{H}_P$ and $\underline{G}_P$ are ideals of $\underline{F}_P$, $\underline{N}_P$ is an ideal of $\underline{G}_P$ and $\underline{G}_P/\underline{N}_P$ is an abelian. The sets G_P and F_P are closed subgroups of G whose Lie algebras are $\underline{G}_P$ and $\underline{F}_P$.*

Proof. Since $\underline{N}_P, \underline{H}_P, \underline{G}_P$ and $\underline{F}_P$ are vector subspaces of $\underline{G}$, it suffices to prove that $[\underline{F}_P, \underline{F}_P] \subset \underline{F}_P$, $[\underline{F}_P, \underline{H}_P] \subset \underline{H}_P$, and $[\underline{G}_P, \underline{F}_P] \subset \underline{N}_P$.

Let $f \in \underline{F}_P$, $g \in \underline{G}_P$ and $\alpha \in P$.

For all $\beta \in \underline{G}^*$, $X \in \underline{G}$ we have $L_X\beta = i_X\, \mathrm{d}\beta$. Then

$$i_{[g,f]}\, \mathrm{d}\alpha = L_{[g,f]}\alpha = L_g L_f \alpha - L_f L_g \alpha$$

and this is the zero 1-form since L_g vanishes on the elements of $\langle P \rangle$.

Moreover,

$$i_{[g,f]}\alpha = L_g i_f \alpha - i_f L_g \alpha$$

vanishes since $i_f\alpha$ is constant and $L_g\alpha = 0$.

This proves that $[\underline{G}_P, \underline{F}_P] \subset \underline{N}_P$. The proofs that $[\underline{F}_P, \underline{F}_P] \subset \underline{F}_P$ and $[\underline{F}_P, \underline{H}_P] \subset \underline{H}_P$ are similar.

For each $\alpha \in P$ let us consider the isotropy subgroup at α corresponding to the action of G on $\underline{G}^*$ defined by the coadjoint representation. Each one of these subgroups of G is closed and its intersection coincides with G_P. It follows that G_P is a closed subgroup.

The subset F_P of G is composed by the g such that Ad_g^* invaries $\langle P \rangle$. It is thus obvious that F_P is a subgroup of G. The fact of F_P being closed can be derived as follows.

Let $L = \{\alpha^1, \ldots, \alpha^r\}$ be a maximal linearly independent subset of P. Then L is a basis of $\langle P \rangle$. Complete L to a basis, $B = \{\alpha^1, \ldots, \alpha^r \beta^{r+1}, \ldots, \beta^n\}$ of $\underline{G}^*$. Then F_P is composed by the $g \in G$ such that the entries (i, j) of the matrix of Ad_g^* in the basis B are 0 whenever $i = r+1, \ldots, n$, $j = 1, \ldots, r$. This proves that F_P is closed.

Let us denote by $\langle P \rangle^\perp$, the subspace of $\underline{G}$ composed by the X such that $\alpha(X) = 0$ for all α in $\langle P \rangle$.

The Lie algebra of F_P, $\underline{F_P}$ is composed by the $X \in \underline{G}$ such that $\mathrm{Exp}\, tX \in F_P$, i.e. $\mathrm{Ad}^*_{\mathrm{Exp\,tX}}\alpha \in \langle P \rangle$ for all $\alpha \in \langle P \rangle$, $t \in \mathbb{R}$.

Thus $X \in \underline{F_P}$ if and only if for all $\alpha \in \langle P \rangle$ and $Y \in \langle P \rangle^\perp$, we have

$$0 = \mathrm{Ad}^*_{\mathrm{Exp}\, tX}\alpha(Y) = \alpha(\mathrm{Ad}_{\mathrm{Exp}(-tX)}(Y)) = \alpha(\mathrm{Exp}(\mathrm{ad}(-tX))(Y)) = \alpha(\mathrm{e}^{-t\, \mathrm{ad}_X}(Y)).$$

As a consequence, the elements of $\underline{F_P}$ are the X such that $\mathrm{e}^{-t\, \mathrm{ad}_X}$ stabilizes $\langle P \rangle^\perp$. This condition is obviously equivalent to the fact that ad_X stabilizes $\langle P \rangle^\perp$.

On the other hand, we have $X \in \underline{F}_P$ if and only if

$$0 = i_X\, \mathrm{d}\alpha(Y) = \mathrm{d}\alpha(X, Y) = \alpha([Y, X]) = -\alpha(\mathrm{ad}_X(Y))$$

for all $\alpha \in \langle P \rangle$ and $Y \in \langle P \rangle^{\perp}$. Thus $\underline{F}_P$ is, like $\underline{F_P}$, composed by the X such that ad_X stabilizes $\langle P \rangle^{\perp}$.

To prove that the Lie algebra of G_P is $\underline{G}_P$ one can use similar arguments, or proceed as follows.

For all $\alpha \in \underline{G}^*$ and $X \in \underline{G}$ we have $\mathrm{Ad}^*_{\mathrm{Exp}\, tX}\alpha = (R_{\mathrm{Exp}\, tX})^*\alpha$, and since the flow of X is $\{R_{\mathrm{Exp}\, tX}\colon \tau \in \mathbb{R}\}$, it follows that

$$R^*_{\mathrm{Exp}\, tX} L_X \alpha = \frac{\mathrm{d}}{\mathrm{d}t}(R_{\mathrm{Exp}\, tX})^*\alpha = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}^*_{\mathrm{Exp}\, tX}\alpha$$

Since the Lie algebra of G_P is composed by the $X \in \underline{G}$ such that $\mathrm{Ad}^*_{\mathrm{Exp}\, tX}\alpha = \alpha$ for all $\alpha \in \langle P \rangle$, $t \in \mathbb{R}$, it follows from the preceding formula that it coincides with $\underline{G}_P$. □

Now, we return to the consideration of the set $Q = \{\underline{A}^*\Omega^1, \ldots, \underline{A}^*\Omega^r\}$.

Let $\Sigma : U \to G$ be a local cross-section of $\underline{a}$ defined in an open neighborhood of a.

Lemma 3.5. *We have*

$$\underline{A} \circ \Sigma = (\Sigma^*\underline{A}^*\Omega^1, \ldots, \Sigma^*\underline{A}^*\Omega^r).$$

Proof. First, notice that if $\omega = (\omega^1, \ldots, \omega^r)$ is a local cross-section of $B\rho$, we have

$$\omega^*\Omega^i = \omega^i$$

for all $i = 1, \ldots, r$. In fact, for all p in the domain of ω and $v \in T_pM$ we have

$$(\omega^*\Omega^i)_p \cdot v = \Omega^i_{\omega(p)}(T_p\omega \cdot v) = \omega^i(p)(T_{\omega(p)}B\rho \circ T_p\omega \cdot v) = \omega^i(p) \cdot v.$$

Thus, since $\underline{A} \circ \Sigma$ is a local cross-section of $B\rho$ we have

$$\Sigma^*\underline{A}^*\Omega^i = (\underline{A} \circ \Sigma)^*\Omega^i = (\underline{A} \circ \Sigma)^i$$

and the result follows. □

Corollary 3.6. *The subset Q of $\underline{G}^*$ is linearly independent.*

Proof. Let $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ be such that

$$\sum_{i=1}^{r} \lambda_i(\underline{A}^*\Omega^i) = 0.$$

Then

$$0 = \Sigma^*\left(\sum_{i=1}^{r} \lambda_i(\underline{A}^*\Omega^i)\right) = \sum_{i=1}^{r} \lambda_i(\Sigma^*\underline{A}^*\Omega^i).$$

But, as a consequence of Lemma 3.5, the $(\Sigma^*\underline{A}^*\Omega^i)_a$ are the elements of the basis $\Sigma \circ \underline{A}(a)$ of $\rho^{-1}(a)$. It follows that $\lambda^i = 0$ for all $i = 1, \ldots, r$. □

Since the $(\Sigma^* \underline{A}^* \Omega^i)_a$, compose a basis of $\rho^{-1}(a)$, we can state for future reference the following result.

Corollary 3.7. *We have*

$$\rho^{-1}(a) = \{(\Sigma^* \sigma)_a \colon \sigma \in \langle Q \rangle\}.$$

Also we have the following proposition.

Proposition 3.8. *The isotropy subgroup at a, G_a, is a Lie subgroup of F_Q whose Lie algebra is contained in $\underline{H}_Q$. The isotropy subgroup at A, G_A, is a closed invariant Lie subgroup of $G_a \cap G_Q$ whose Lie algebra is contained in N_Q.*

Proof. Let $g \in G_a$, $Y \in \underline{G}$ and $i \in \{1, \ldots, r\}$. Then we have $\underline{a} \circ R_g = \underline{a}$ and

$$\begin{aligned}
\mathrm{Ad}_g^*(\underline{A}^* \Omega^i) \cdot Y &= R_g^* \underline{A}^* \Omega^i \cdot Y = ((\underline{A} \circ R_g)^* \Omega^i)_e \cdot Y_e \\
&= \Omega^i_{g_{BS}(A)}(T_e(\underline{A} \circ R_g) \cdot Y_e) \\
&= (g_M^{-1})^* \alpha^i (T_{g_{BS}(A)} B\rho \circ T_e(\underline{A} \circ R_g) \cdot Y_e) \\
&= \sum_{j=1}^r s_i^j(g) \alpha^j (T_g(B\rho \circ \underline{A}) \circ T_e R_g \cdot Y_e) \\
&= \sum_{j=1}^r s_i^j(g) \alpha^j (T_g \underline{a} \circ T_e R_g \cdot Y_e) = \sum_{j=1}^r s_i^j(g) \alpha^j (T_e(\underline{a} \circ R_g) \cdot Y_e) \\
&= \sum_{j=1}^r s_i^j(g) \alpha^j (T_e \underline{a} \cdot Y_e)
\end{aligned}$$

where $s_i^j(g)$ is the (j, i) entry of $s(g)$.

If we take $g = e$ in this formula, we obtain

$$\alpha^i (T_e \underline{a} \cdot Y_e) = \underline{A}^* \Omega^i \cdot Y,$$

so that

$$\mathrm{Ad}_g^*(\underline{A}^* \Omega^i) \cdot Y = \sum_{j=1}^r s_i^j(g) \underline{A}^* \Omega^j \cdot Y = \left(\sum_{j=1}^r s_i^j(g) \underline{A}^* \Omega^j \right) \cdot Y.$$

This proves that $G_a \subset F_Q$.

Now, we must prove that $\underline{A}^* \Omega^i \cdot \nu = 0$ for all $i = 1, \ldots, r$, $\nu \in \underline{G_a}$.

Let $\Sigma : U \to G$ be a local cross-section of $\underline{a}$, where U is an open neighborhood of a, and let $h : U \to G_a$ be a C^∞ map such that $h(a) = e$. The map $\Sigma' : U \to G$ given by $\Sigma'(u) = \Sigma(u)\, h(u)$ is also a section of $\underline{a}$. As a consequence of Lemma 3.5,

$$\begin{aligned}
((\Sigma'^* \underline{A}^* \Omega^1)_a, \ldots, (\Sigma'^* \underline{A}^* \Omega^r)_a) &= \underline{A} \circ \Sigma'(a) \\
&= \underline{A} \circ \Sigma(a) = ((\Sigma^* \underline{A}^* \Omega^1)_a, \ldots, (\Sigma^* \underline{A}^* \Omega^r)_a)
\end{aligned}$$

Now, let $\gamma(t)$ be a differentiable curve in M such that $\gamma(0) = a$. Then we have

$$\begin{aligned}
&(\Sigma'^{*}\underline{A}^{*}\Omega^{i})_{a}\left(\dot{\overline{\gamma(t)}}\right)\\
&= (\underline{A}^{*}\Omega^{i})_{\Sigma'(a)}\left(\dot{\overline{\Sigma(\gamma(t))h(\gamma(t))}}\right)\\
&= (\underline{A}^{*}\Omega^{i})_{\Sigma(a)}\left(\dot{\overline{\Sigma(\gamma(t))}} + T_{e}L_{\Sigma(a)}\left(\dot{\overline{h(\gamma(t))}}\right)\right)\\
&= (\underline{A}^{*}\Omega^{i})_{\Sigma(a)}\left(\dot{\overline{\Sigma(\gamma(t))}}\right) + ((L_{\Sigma(a)})^{*}(\underline{A}^{*}\Omega^{i}))_{e}\left(\dot{\overline{h(\gamma(t))}}\right)\\
&= (\Sigma^{*}\underline{A}^{*}\Omega^{i})_{a}\left(\dot{\overline{\gamma(t)}}\right) + (h^{*}\underline{A}^{*}\Omega^{i})_{a}\left(\dot{\overline{\gamma(t)}}\right).
\end{aligned}$$

It follows that $(h^{*}\underline{A}^{*}\Omega^{i})_{a}\left(\dot{\overline{\gamma(t)}}\right) = 0$, i.e. $(h^{*}\underline{A}^{*}\Omega^{i})_{a} = 0$. This relation, for h an arbitrary differentiable function with values in G_a such that $h(a) = e$, imply $\underline{A}^{*}\Omega^{i}|_{\underline{G_a}} = 0$, i.e. $\underline{G_a} \subset \underline{H}_Q$.

We already know that G_A is a closed invariant subgroup of G_a. If $g \in G_A$ we have $\underline{A} \circ R_g = \underline{A}$. Then $\mathrm{Ad}_{g}^{*}\underline{A}^{*}\Omega^{i} = R_{g}^{*}\underline{A}^{*}\Omega^{i} = \underline{A}^{*}\Omega^{i}$ so that $g \in G_P$. Therefore, $G_A \subset G_a \cap G_P$, and as a consequence, $\underline{G_A} \subset \underline{H}_P \cap \underline{G}_P = \underline{N_P}$. □

To end this section, let us assume that G is a transitive stability subgroup. As a consequence of Proposition 3.8 and Corollary 3.7 we see that there exist a subset of $\underline{G}^{*}$, P, and a closed subgroup of F_P whose Lie algebra is in $\underline{H}_P$, H, such that

- M is, up to an equivariant diffeomorphism, G/H.
- The fibre of S at $p \in G/H$ is $\{(\Sigma^{*}\sigma)_p : \sigma \in \langle P\rangle\}$, where Σ is a local cross-section of the canonical map from G onto G/H, defined in a neighborhood of p.

In the next section, we prove that any pair (P, H) as above leads to a Pfaff system admitting G as a transitive stability subgroup.

4. Construction of Pfaff systems with a given transitive stability subgroup

Let G be a Lie group, P a subset of $\underline{G}^{*}$, $\{\sigma^{1}, \ldots, \sigma^{r}\}$ a maximal linearly independent subset of P and H a closed subgroup of F_P whose Lie algebra, $\underline{H}$, is contained in $\underline{H}_P$. We denote by $\pi_H : G \to G/H$ the canonical map.

Let $p \in G/H$ and Σ a local cross-section of π_H defined in a neighborhood of p. Let us denote by S_p the set composed by the $\Sigma^{*}\sigma$ such that $\sigma \in \langle P\rangle$.

Lemma 4.1. *The set S_p does not depend on Σ.*

Proof. Let Σ' be another section of π_H defined in a neighborhood of p. In the intersection of the domains of Σ and Σ', U, there exist a uniquely defined differentiable function with values in H , h, such that $\Sigma'(q) = \Sigma(q)h(q)$ for all $q \in U$.

We shall prove that

$$(\Sigma'^{*}\sigma)_p = (\Sigma^{*}(\mathrm{Ad}_{h(p)}^{*}\sigma))_p \tag{4.1}$$

for all $\sigma \in \langle P\rangle$.

This equation can be used to prove the lemma as follows. Since $h(p) \in H \subset F_P$, Eq. (4.1) entails

$$\{(\Sigma'^{*}\sigma)_p \colon \sigma \in \langle P\rangle\} \subset \{(\Sigma^{*}\sigma)_p \colon \sigma \in \langle P\rangle\}.$$

One thus sees that the sets are equal by interchanging the roles of Σ and Σ'.

In order to prove (4.1) we consider a differentiable curve in M, $\gamma(t)$ such that $\gamma(0) = p$. The following computations are more or less straightforward.

$$\begin{aligned}
&(\Sigma'^{*}\sigma)_p\left(\overline{\dot{\gamma(t)}}\right) = \sigma_{\Sigma'(p)}\left(\overline{\Sigma'(\gamma(t))}^{\cdot}\right)\\
&= \sigma_{\Sigma(p)h(p)}\left(\overline{\Sigma(\gamma(t))h(\gamma(t))}^{\cdot}\right)\\
&= \sigma_{\Sigma(p)h(p)}\left(T_{h(p)}L_{\Sigma(p)}\left(\overline{h(\gamma(t))}^{\cdot}\right) + T_{\Sigma(p)}R_{h(p)}\left(\overline{\Sigma(\gamma(t))}^{\cdot}\right)\right)\\
&= (L^{*}_{\Sigma(p)}\sigma)_{h(p)}\left(\overline{h(\gamma(t))}^{\cdot}\right) + (R^{*}_{h(p)}\sigma)_{\Sigma(p)}\left(\overline{\Sigma(\gamma(t))}^{\cdot}\right)\\
&= \sigma_{h(p)}\left(\overline{h(\gamma(t))}^{\cdot}\right) + (\mathrm{Ad}^{*}_{h(p)}\sigma)_{\Sigma(p)}\left(T_p\Sigma\cdot\overline{(\gamma(t))}^{\cdot}\right)\\
&= \sigma_{h(p)}\left(\overline{h(\gamma(t))}^{\cdot}\right) + (\Sigma^{*}(\mathrm{Ad}^{*}_{h(p)}\sigma))_p\left(\overline{\gamma(t)}^{\cdot}\right)
\end{aligned}$$

and (4.1) follows from the fact that $\overline{h(\gamma(t))}^{\cdot}$ is tangent to H, so that the corresponding left invariant vector field is in $\underline{H}$ and σ vanishes on $\underline{H}$. □

Lemma 4.2. *We have* $\dim\, S_p = r$ *for all* $p \in G/H$.

Proof. Since $\{(\Sigma^{*}\sigma^1)_p, \ldots, (\Sigma^{*}\sigma^r)_p\}$ is a set of generators of S_p, it suffices to prove that they are linearly independent.

Let $\lambda^1, \ldots, \lambda^r \in \mathbb{R}$ be such that $\sum_{i=1}^{r} \lambda^i(\Sigma^{*}\sigma^i)_p = 0$. Then $\Sigma^{*}(\sum_{i=1}^{r}\lambda^i\sigma^i)_p = 0$, so that $\sum_{i=1}^{r}\lambda^i\sigma^i$ vanishes on $T_p\Sigma(T_p(G/H))$. But the tangent space of G at $\Sigma(p)$ is a direct sum of $T_p\Sigma(T_p(G/H))$ and the tangent space at $\Sigma(p)$ to the submanifold $\Sigma(p)H$, which is given by the values at $\Sigma(p)$ of the left invariant vector fields contained in $\underline{H}$. Since $\sum_{i=1}^{r}\lambda^i\sigma^i$ vanishes on $\underline{H}$, it follows that $(\sum_{i=1}^{r}\lambda^i\sigma^i)_{\Sigma(p)} = 0$. Then $\sum_{i=1}^{r}\lambda^i\sigma^i = 0$, so that $\lambda^i = 0$, $i = 1, \ldots, r$. □

Let $S = \bigcup_{p\in G/H} S_p$ and let ρ be the restriction to S of the canonical projection of $T^{*}(G/H)$ onto G/H.

Each local cross-section, $\Sigma : U \to G$, of π_H, gives rise to a set of sections of ρ, $\{\Sigma^{*}\sigma^1, \ldots, \Sigma^{*}\sigma^r\}$, defined on U. Thus we define

$$\Psi_\Sigma : (p, (\lambda^1, \ldots, \lambda^r)) \in U \times \mathbb{R}^r \to \sum_{i=1}^{r} \lambda^i(\Sigma^{*}\sigma^i)_p \in \rho^{-1}(U).$$

There exists an unique topology and an unique differentiable structure on S such that all the Ψ_Σ are diffeomorphisms onto its image. When one considers on S this differentiable structure, ρ becomes a vector bundle, having the Ψ_Σ as trivializations.

Now let us denote by λ the canonical injection of S into $T^*(G/H)$. Since each $\Sigma^*\sigma^i$ is a differentiable section of τ^*, it follows that $\lambda \circ \Psi_\Sigma$ is differentiable so that $\lambda|_{\rho^{-1}(U)}$ is C^∞. Then λ is C^∞ and thus an injective homomorphism of vector bundles.

Finally, let us see that the diffeomorphisms of G/H associated to the canonical action stabilizes S.

Let $g \in G$, $\sigma \in \langle P\rangle$, (U, Σ) be a section of π_H and $p \in U$. Then, we have

$$\begin{aligned}(g_{G/H}^{-1})^*(\Sigma^*\sigma)_p &= ((\Sigma \circ g_{G/H}^{-1})^*\sigma)_{g_{G/H}(p)} \\ &= ((\Sigma \circ g_{G/H}^{-1})^* L_g^*\sigma)_{g_{G/H}(p)} = ((L_g \circ \Sigma \circ g_{G/H}^{-1})^*\sigma)_{g_{G/H}(p)},\end{aligned}$$

and since $L_g \circ \Sigma \circ g_{G/H}^{-1}$ is also a section of π_H, it follows that

$$((L_g \circ \Sigma \circ g_{G/H}^{-1})^*\sigma)_{g_{G/H}(p)}$$

is in S.

5. Case where the Pfaff system is given by globally defined forms

Let (ρ, λ) be a Pfaff system generated in the way explained in the preceding section, by a subset, P, of $\underline{G}^*$ and a closed subgroup, H, of $\underline{F}_P$ whose Lie algebra, $\underline{H}$, is contained in $\underline{H}_P$.

This Pfaff system can be obtained from a set of r globally defined 1-forms, as indicated in Section 1, if and only if the principal fibre bundle of the basis of S admits a globally defined differentiable cross-section. In fact, if $(\omega^1, \dots, \omega^r)$ is a global differentiable cross-section of $B\rho$, for all $g \in G$, the 1-form $g_{G/H}^*\omega^i$ is a section of ρ so that there exist differentiable functions, g_j^i, on G/H such that

$$g_{G/H}^*\omega^i = \sum_{j=1}^{r} g_j^i\omega^j$$

for all $i = 1, \dots, r$. The converse is trivial.

In what follows all cross-sections and functions are supposed to be differentiable unless otherwise stated.

A sufficient condition for the existence of a global cross-section of $B\rho$ is the existence of a global cross-section of π_H. In fact, if $\Sigma : G/H \to G$ is a global cross-section of π_H, $(\Sigma^*\sigma^1, \dots, \Sigma^*\sigma^r)$ is a global cross-section of $B\rho$. In Proposition 5.2 we give a weaker sufficient condition.

Let $A \in B\rho^{-1}(H)$. We have seen in Section 3 that the isotropy subgroup at A, G_A, is a normal subgroup of the isotropy subgroup at H, H. Hence $(G/G_A)(G/H, H/G_A)$ is a principal fibre bundle. The bundle projection is the map

$$\underline{\underline{\pi_H}} : g\, G_A \in G/G_A \to g\, H \in G/H.$$

In the same way as in Section 3, we define a representation, s_A, of H in $GL(r, \mathbb{R})$ by means of $h_{BS}(A) = A \star s_A(h)$ for all $h \in H$ and we see that G_A is the kernel of this representation.

Thus s_A defines, in the well-known way, an injective homomorphism, $\underline{\underline{s}}$, from H/G_A into $GL(r, \mathbb{R})$.

Let $\underline{\underline{A}}$ be the canonical immersion

$$\underline{\underline{A}} : g\, G_A \in G/G_A \rightarrow g_{BS}(A) \in BS.$$

Lemma 5.1. *The pair* ($\underline{\underline{A}}$, $\underline{\underline{s}}$) *is an injective homomorphism of principal fibre bundles.*

Proof. We only need to prove that $\underline{\underline{A}}(gG_A \star hG_A) = \underline{\underline{A}}(gG_A) \star \underline{\underline{s}}(hG_A)$ for all $g \in G$, $h \in H$, where $\star$ means the action of the corresponding structural group in each side. But we have

$$\begin{aligned}\underline{\underline{A}}(g\, G_A \ \star\ h\, G_A) &= \underline{\underline{A}}(g\, h\, G_A) = (g\, h)_{BS}(A) = A \ \star\ s_A(g\, h)\\ &= (A \ \star\ s_A(g)) \ \star\ s_A(h) = (g_{BS}(A)) \ \star\ \underline{\underline{s}}(h\, G_A)\\ &= \underline{\underline{A}}(g\, G_A) \ \star\ \underline{\underline{s}}(h\, G_A). \qquad \Box\end{aligned}$$

As a consequence, the existence of global sections of $\underline{\underline{\pi_H}}$ entails the existence of global sections of $B\rho$.

In Proposition 5.2 we use this fact to give a sufficient condition that can be stated directly in terms of the given data P and H, and needs no choice of a basis, A, in $B\rho^{-1}(H)$.

There exists another representation of H in $GL(r, \mathbb{R})$ associated to each maximal linearly independent subset of P. In fact, if $B = (\sigma^1, \ldots, \sigma^r)$ is a maximal linearly independent subset of P and $h \in H$, there exists $(k_B)^i_j(h) \in \mathbb{R}$ such that

$$\mathrm{Ad}^*_h(\sigma^i) = \sum_{j=1}^{r} (k_B)^j_i(h)\, \sigma^j$$

for all $i = 1, \ldots, r$. Then, if we denote by $k_B(h)$ the matrix whose $(j,\ i)$ entry is $(k_B)^j_i(h)$, the map, k_B, defined by sending h to $k_B(h)$ is a representation.

Proposition 5.2. *If the canonical map from* $G/\mathrm{Ker}\, k_B$ *onto* G/H *has a global cross-section, then the Pfaff system determined by* P *and* H *can be given by globally defined 1-forms.*

Proof. Let $\Sigma : U \rightarrow G$ be a local cross-section of π_H such that $\Sigma(H) = e$, where U is an open neighborhood of H, and let B as above. Let us denote $((\Sigma^*\sigma^1)_H, \ldots, (\Sigma^*\sigma^1)_H) \in B\rho^{-1}(H)$ by A.

We only need to prove that $s_A = k_B$.

Let $h \in H$. The map $L_h \circ \Sigma \circ h^{-1}_{G/H}$ is a section of π_H defined in $h_{G/H}(U)$, which is an open neighborhood of H. Let $U' = U \cap h_{G/H}(U)$ and let f be the function with values in H such that $\Sigma' = \Sigma\, f$ (see the Proof of Lemma 4.1). We have

$$f(H) = \Sigma(H)\, f(H) = \Sigma'(H) = h\, \Sigma(h^{-1}\, H) = h$$

Thus formula (4.1) tells us that

$$(\Sigma'^{*}\sigma^{i})_{H} = (\Sigma^{*}(\mathrm{Ad}^{*}_{f(H)}\sigma^{i}))_{H} = (\Sigma^{*}(\mathrm{Ad}^{*}_{h}\sigma^{i}))_{H} \tag{5.1}$$

and we obtain

$$(h^{-1}_{G/H})^{*}(\Sigma^{*}\sigma^{i})_{H} = (\Sigma'^{*}\sigma^{i})_{H} = (\Sigma^{*}(\mathrm{Ad}^{*}_{h}\sigma^{i}))_{H} = \sum_{j=1}^{r} (k_B)^{j}_{i}(h)\,(\Sigma^{*}\,\sigma^{j})_{H}.$$

Then we have

$$A \star k_B(h) = ((h^{-1}_{G/H})^{*}(\Sigma^{*}\,\sigma^{1})_{H}, \ldots, (h^{-1}_{G/H})^{*}(\Sigma^{*}\,\sigma^{r})_{H}) = h_{BS}(A)$$

so that $s_A(h) = k_B(h)$. □

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